LEVEL OPTIMIZATION IN THE TOTALLY REAL CASE

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Abstract. In this paper, congruences between holomorphic Hilbert modular forms are studied. We show the best possible level optimization result outside ℓ for $\ell \geq 3$ by solving the remaining case of Mazur principle when the degree of the totally real field is even.

1. Introduction

The aim of this paper is to discuss mod ℓ congruences between Hilbert modular forms. Here ℓ is a fixed prime. More precisely, given a holomorphic Hilbert modular form f of some level and weight, we look for another modular form g with a smaller level and the same weight whose Fourier coefficient are congruent to that of f. For elliptic modular forms, i.e., when $F = \mathbb{Q}$, such studies have been done by J. P. Serre, B. Mazur, K. A. Ribet, F. Diamond, and by other authors(see the references of [19]). Amazingly, to obtain some optimal g, the correct condition imposed on f involves an information from the Galois representation attached to f, so we need to use Galois representations to study congruences.

We use a representation theoretical terminology in the adelic setting, since it is essential in the local analysis. For a totally real field F, let $I_F = \{\iota : F \hookrightarrow \mathbb{R}\}$ be the set of the infinite places of F. We take an element $k = (k_\iota)_{\iota \in I_F} \in \mathbb{Z}^{I_F}$ and $w \in \mathbb{Z}$, where $k_\iota \geq 2$, and $k_\iota \equiv w \mod 2$ for all ι. Let π be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$ having infinity type (k, w) (see notations for our normalization). Those types of cuspidal representations are generated by holomorphic Hilbert cusp forms.

Fix a prime ℓ, and an isomorphism $\mathbb{C} \simeq \bar{\mathbb{Q}}_\ell$ by the axiom of choice. It is known that the finite part π^∞ of π is defined over some algebraic number field, and hence over some finite extension E_λ of $\mathbb{Q}_\ell$. In this case we say that π is defined over E_λ. For the integer ring $\mathscr{O}_\lambda$ of E_λ, there is a two dimensional continuous λ-adic Galois representation

$$\rho_{\pi,\lambda} : G_F \to \mathrm{GL}_2(\mathscr{O}_\lambda)$$

associated to π (see [17], [5], [22], [21], [1] for $F \neq \mathbb{Q}$). Here $G_F = \mathrm{Gal}(\bar{F}/F)$ is the absolute Galois group of F. Two cuspidal representations π and π' which are defined over some E_λ are *congruent* mod λ if the semi-simplifications of the mod λ-reductions of their λ-adic representations are isomorphic over k_λ:

$$(\rho_{\pi,\lambda} \mod \lambda)^{\mathrm{ß}} \simeq (\rho_{\pi',\lambda} \mod \lambda)^{\mathrm{ß}}.$$

Here $(-)^{\mathrm{ß}}$ denotes the semi-simplification. We note that this notion of congruences is equivalent to the congruences between the Fourier coefficients of the corresponding normalized Hilbert modular newforms by the Chebotarev density theorem.

We call an absolutely irreducible mod ℓ continuous representation $\bar{\rho} : G_F \to \mathrm{GL}_2(\bar{k}_\lambda)$ *modular* if it is isomorphic to some $\rho_{\pi,\lambda} \mod \lambda$ over $\bar{k}_\lambda$. Even if $\bar{\rho}$ is modular, there are many cuspidal representations giving the same $\bar{\rho}$, which are all seen as *deformations* of $\bar{\rho}$ from the viewpoint of Mazur, and the purpose of this paper is restated as to find a good cuspidal representation π' which is optimal for modular representation $\bar{\rho}$ in a suitable sense.

As is already mentioned, $F = \mathbb{Q}$ case is studied well, so we restrict our attention to general totally real F other than $\mathbb{Q}$.

Here is our main theorem, which treats the level optimization at $v \not| \ell$.

Theorem 1.1. *[Theorem A] Let $\bar{\rho} : G_F \to \mathrm{GL}_2(\bar{k}_\lambda)$ be a continuous absolutely irreducible mod ℓ-representation satisfying A1)-A3):*

A-1) *$\ell \geq 3$, and $\bar{\rho}|_{F(\zeta_\ell)}$ is absolutely irreducible if $[F(\zeta_\ell) : F] = 2$.*
A-2) *$\bar{\rho} \simeq \rho_{\pi,\lambda} \mod \lambda$. Here π is a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of infinity type (k, w) defined over E_λ, satisfying $(\pi^\infty)^K \neq \{0\}$ for some compact open subgroup $K = \prod_u K_u$ of $\mathrm{GL}_2(\mathbb{A}_F^\infty)$,*
A-3) *for a place $v \not| \ell$, $\bar{\rho}$ is either ramified at v, or $q_v \equiv 1 \mod \ell$.*

Then there is a cuspidal representation π', having the same infinity type (k, w) as π, defined over a finite extension $E'_{\lambda'} \supset E_\lambda$ such that the following conditions hold:

1. *The associated λ'-adic representation $\rho_{\pi',\lambda'}$ gives $\bar{\rho}$. $({\pi'}^{v,\infty})^{K^v} \neq \{0\}$,*
2. *The conductor $\mathrm{cond}(\pi'_v)$ of π'_v is equal to $\mathrm{Art}\bar{\rho}|_{G_v}$, where G_v is the decomposition group at v, and $\mathrm{Art} \in \mathbb{Z}$ means the Artin conductor,*
3. *$\det\rho_{\pi',\lambda'} \cdot \chi_{\mathrm{cycle}}^{w+1}$ is the Teichmüller lift of $\det\bar{\rho} \cdot \bar{\chi}_{\mathrm{cycle}}^{w+1}$.*

If π' gives $\bar{\rho}$, we always have the basic inequality

$$\mathrm{cond}\pi_v \geq \mathrm{Art}\bar{\rho}|_{G_v},$$

so the equality in 1.1 (2) is optimal. When we remove condition A-3), i.e., even when $\bar{\rho}$ is unramified at v and $q_v \equiv 1 \mod \ell$, we still have some π' giving $\bar{\rho}$ with the property that π'_v is spherical, or a special representation twisted by an unramified character. This missing case in theorem A, i.e., the case when $\bar{\rho}$ is unramified at v and $q_v \equiv 1 \mod \ell$, was treated by K. A. Ribet in case of $\mathbb{Q}$ [18], and by A. Rajaei in the totally real case.

Especially, we note that theorem A includes

Corollary 1.2 (Corollary A' (Mazur principle)). *There exists π' as in theorem A when $\bar{\rho}$ is unramified at v, and $q_v \not\equiv 1 \mod \ell$.*

We should note that theorem A is a stronger form of theorem B below, which was obtained earlier by F. Jarvis [14], [15] (some additional condition on $\bar{\rho}$ is put in the references compared to theorem B, but it is easily removed, see lemma 4.1 in 4.1.

Theorem 1.3 (Theorem B (Jarvis)). *In addition to assumptions A-1)-A-3) of theorem A, we assume*

A-4) *If the degree $[F : \mathbb{Q}] = g$ is even, assume that π_u is essentially square integrable for some $u \neq v$, $v \not| \ell$, with $K_u = K_1(m_u^{\mathrm{cond}\pi_u})$.*

Then we have the same conclusion as in theorem A.

Condition A-4 in theorem B especially excludes the unramified case when the degree $[F : \mathbb{Q}]$ is even. Logically, theorem A is a consequence of theorem B and corollary A' in the full form. Corollary A' in the even degree case, i.e., when $[F : \mathbb{Q}]$ is even, is one of the main contributions of this paper. We also give a new proof to theorem B in this article.

Let us explain the method of proofs. In [14], Jarvis proved a part of corollary A', i.e., the Mazur principle under A-1)-A-4), by a detailed study on the arithmetic models of Shimura curves. Note that assumption A-4) in theorem B is used to relate $\bar{\rho}$ to the cohomology of a Shimura curve associated with a division algebra which is *unramified* at v. When $\bar{\rho}$ is ramified at v ([15]), Jarvis used the argument of Carayol in [6], which does not use any arithmetic models. The methods are cohomological.

Our corollary A', especially the Mazur principle in the even degree case, has been thought difficult since the λ-adic representation $\rho_{\pi,\lambda}$ may not be obtained from a Shimura curve in general [21]. Surprisingly, we can use a Shimura curve attached to a division quaternion algebra which is *ramified* at v in our proof.

Our method to prove corollary A', and to give a new proof to theorem B, is also cohomological and summarized in the following way.

For any finite $\mathscr{O}_\lambda$-algebra R, we define a cohomological functor H_R from Shimura curves, on which G_F acts, and behaves nicely under any scalar extension $R \to R'$. $H_{\mathscr{O}_\lambda} \otimes_{\mathscr{O}_\lambda} E_\lambda$ consists of representations $\rho_{\pi',\lambda'}$'s giving $\bar{\rho}$. The main observation is that the inertia fixed part $H_R^{I_v}$ also commutes with scalar extensions, which shows theorem B when $\bar{\rho}$ is ramified at v, since we have some π' with $\rho_{\pi',\lambda'}^{I_v} \neq \{0\}$ under the assumption $\bar{\rho}^{I_v} \neq \{0\}$. The equality of Artin conductors follows easily from this.

To show the good property of $H_R^{I_v}$ with respect to scalar extensions in case of theorem B, we use a deep arithmetic geometrcal result, namely the regularity of the arithmetic models of Shimura curves using Drinfeld level structures. By this result and the purity theorem of Zariski-Nagata for étale coverings of a regular scheme, we analyze the cohomology groups directly, without any calculation of vanishing cycles. Our result is seen as a mod ℓ-version of the local invariant cycle theorem.

In the case of corollary A', the analysis of the cohomology is done with the help of a Cerednik-Drinfeld type theorem obtained by Boutot-Zink [2], based on the principle used for proving theorem B. Also in 4.3, we give an interpretation of Carayol 's lemma [6] by a standard homological algebra. It is an application of a property of perfect complexes, and also cohomological. The methods developed in this article are new even for $F = \mathbb{Q}$. It is our belief that a level optimization in easier situation is a consequence of a homological algebra on Shimura varieties. We hope that the method in this paper is effective in some higher dimensional cases as well. One may use O. Gabber's absolute purity theorem ([9]) instead of the purity theorem of Zariski-Nagata.

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2. Notations

For a number field F, $\mathscr{O}_F$ is the integer ring, and $G_F = \mathrm{Gal}(\bar{F}/F)$ is the absolute Galois group. For a place v of F, F_v means the local field at v. For a finite place v, o_v is the integer ring of F_v, with the maximal ideal m_v, $k(v) = o_v/m_v$ is the residue field with the cardinal q_v. p_v is a uniformizer of m_v. G_v and I_v mean the decomposition and the infertia groups at place v, respectively.

$\mathbb{A}_F$ means the adéle ring of F, $\mathbb{A}_F^\infty$, and $(\mathbb{A}_F)_\infty$ the finite part and the infinite part, respectively. For a non-zero ideal $\mathfrak{f}$ of $\mathscr{O}_F$, we define compact open subgroups of $\mathrm{GL}_2(\mathbb{A}_F^\infty)$ by

$$K(\mathfrak{f}) = \{g \in \mathrm{GL}_2(\prod_{u:\text{finite}} o_u),\ g \equiv 1 \mod \mathfrak{f}\},$$

$$K_{11}(\mathfrak{f}) = \{g \in \mathrm{GL}_2(\prod_{u:\text{finite}} o_u),\ g \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod \mathfrak{f}\},$$

$$K_1(\mathfrak{f}) = \{g \in \mathrm{GL}_2(\prod_{u:\text{finite}} o_u),\ g \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \mod \mathfrak{f}\},$$

$$K_0(\mathfrak{f}) = \{g \in \mathrm{GL}_2(\prod_{u:\text{finite}} o_u),\ g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod \mathfrak{f}\}.$$

We use the similar notation for the corresponding local group.

For an infinity type (k, w), $k \in \mathbb{Z}^I$, $w \in \mathbb{Z}$, which satisfies $k_\iota \equiv w \mod 2$, $k' \in \mathbb{Z}^I$ is defined by the formula

$$k + 2k' = (w + 2) \cdot (1, \ldots, 1).$$

As in the introduction, we fix an isomorphism $\mathbb{C} \simeq \bar{\mathbb{Q}}_\ell$ by the Axiom of Choice. The local Langlands correspondence for $\mathrm{GL}_2(F_v)$ defines a bijection between isomorphism classes of F-semisimple representation ρ_v and admissible representation π_v of $\mathrm{GL}_2(F_v)$.

Our normalization of the local Langlands correspondence is as follows. For the local class field theory, we assume a geometric Frobenius element corresponds to a uniformizer. For a finite place $v \not| \ell$ with spherical π_v, a geometric Frobenius element Fr_v satisfies

$$\mathrm{trace}\rho_{\pi,\lambda}(\mathrm{Fr}_v) = \alpha_v + \beta_v$$

where $(\alpha_v,\ \beta_v)$ is the Satake parameter of π_v seen as a semi-simple conjugacy class in the dual group $\mathrm{GL}_2^\vee(\bar{\mathbb{Q}}_\ell)$, and π_v is a constituant of the non-unitary induction

$$\mathrm{Ind}_{B(F_v)}^{G(F_v)} \chi_{\alpha_v,\beta_v} = \{f : \mathrm{GL}_2(F_v) \to \bar{\mathbb{Q}}_\ell,\ f(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g) = \chi_{\alpha_v,\beta_v}(a, d)|a|_v f(g)\}.$$

Here B is the standard Borel subgroup consisting of the upper triangular matrices,

$$\chi_{\alpha_v,\beta_v} : B(F_v) \to F_v \times F_v \to \bar{\mathbb{Q}}_\ell$$

is the unramified character given by $\chi_{\alpha_v,\beta_v}(a, b) = \alpha_v^{\mathrm{ord}_v a} \beta_v^{\mathrm{ord}_v b}$.

At infinite places, $\mathrm{GL}_2(\mathbb{R})$-representation $D_{k,w}$ corresponds to the *unitarily* induced representation

$$\mathrm{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})} (\mu_{k,w}, \nu_{k,w})_{\mathbf{u}}$$

for two characters of split maximal torus

$$\mu_{k,w}(a) = |a|^{\frac{1}{2}-k'} (\mathrm{sgn} a)^{-w}$$

$$\nu_{k,w}(d) = |d|^{\frac{1}{2}-w+k'}$$

for k' satisfying $k - 2 + 2k' = w$. This normalization, which is the $|\cdot|_v^{\frac{1}{2}}$-twist of unitary normalization, preserves the field of definition. The central character of π corresponds to $\det\rho_{\pi,\lambda}(1)$. Our normalization is basically the same as in [5], except one point. In [5], an arithmetic Frobenius element corresponds to a uniformizer.

The global correspondence $\pi \mapsto \rho_{\pi,\lambda}$ is compatible with the local Langlands correspondence for $v \not| \ell$ ([5] théorème (A), see [22], [21], theorem 2, for the missing even degree cases): the F-semisimplification of $\rho_{\pi,\lambda}|_{G_v}$ corresponds to π_v by the local correspondence normalized as above.

3. Preliminaries

3.1. **Shimura curves and Hida varieties.** We assume $[F:\mathbb{Q}]>1$ in the following (cf. introduction). One can include $\mathbb{Q}$-case with a slight modification. Fix an element $\iota_1 \in I_F$. Take a division quaternion algebra D over F which ramifies at all infinite places other than ι_1, and ramifies possibly at ι_1. We fix an identification

$$D \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_2(\mathbb{R})^{g'} \times \mathrm{H}^{g-g'},$$

where $g'=1$ or 0 according to D is split at ι_1 or not. Here H is the Hamilton quaternion algebra over $\mathbb{R}$. We put

$$G_D = \mathrm{Res}_{F/\mathbb{Q}} D^\times.$$

Here Res means the Weil restriction of scalars. For a compact open subgroup $K \subset D^\times(\mathbb{A}^\infty)$, we define the associated modular variety with a complex structure, by

$$S_K = D^\times \backslash D^\times(\mathbb{A})/K \times K_\infty.$$

Here K_∞ is the maximal compact subgroup of $D^\times(\mathbb{R})$ modulo center. When g' is one, S_K is a Shimura curve, and it has a canonical model $S_{K,F}$ defined over F. When g' is zero, the zero-dimensional variety was used extensively by H. Hida [12] in his study of ℓ-adic Hecke algebras. Note that this is *not* a Shimura variety in the sense of Deligne.

3.2. **λ-adic local systems.** Fix a prime ℓ, and an ℓ-adic field E_λ with the integer ring $\mathscr{O}_\lambda$. We denote the maximal ideal of $\mathscr{O}_\lambda$ by λ and the residue field by k_λ.

For a pair (k,w), $k \in \mathbb{Z}^{I_F}$, $w \in \mathbb{Z}$ as in the introduction, we define an ℓ-adic sheaf $\bar{\mathscr{F}}_{k,w}$ on S_K ([5], p.418-419 for $g'=1$).

Since we need a $\mathbb{Z}_\ell$-structure for this sheaf, we briefly review the construction, assuming D splits at all $v|\ell$, which is sufficient for our later use.

Let $\pi_\ell : \tilde{S}_\ell \to S_{D,K}$ be the Galois covering corresponding to $\prod_{u|\ell} K_u/K \cap \overline{F^\times}$. Here $\overline{F^\times}$ is the closure inside $D^\times(\mathbb{A}_F^f)$.

We choose an isomorphism $D \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq \prod_{v|\ell} M_2(F_v)$. This determines a maximal hyperspecial subgroup of $D^\times(\mathbb{Q}_\ell)$, which we identify with $\prod_{v|\ell} \mathrm{GL}_2(o_v)$. We take E_λ large enough so that all F_v, $v|\ell$ are embeded into E_λ over $\mathbb{Q}_\ell$. So the representation

$$V_{k,w} = \otimes_{\iota \in I_F} (\iota \det)^{-k'_\iota} \, \mathrm{Sym}^{\vee \otimes (k_\iota - 2)}$$

of $D^\times(\mathbb{Q}_\ell)$ is defined over E_λ, and has an $\mathscr{O}_\lambda$-lattice

$$V_{(k,w),\mathscr{O}_\lambda} = \otimes_{v|\ell, \iota : F_v \to E_\lambda} (\iota \det)^{-k'_\iota} \, \mathrm{Sym}^{\vee \otimes (k_\iota - 2)} \, \mathscr{O}_v^{\oplus 2}) \otimes_\iota \mathscr{O}_\lambda.$$

The $\mathscr{O}_\lambda$-smooth sheaf $\bar{\mathscr{F}}^D_{k,w}$ is obtained from the covering π_ℓ and the representation $V_{(k,w),\mathscr{O}_\lambda}$ by contraction. Note that the action of $K \cap \overline{F^\times}$ on the representation $V_{k,w}$ is trivial if K is sufficiently small, so that the sheaf is well-defined.

See [5], p.418-419 for the Betti-version of $\bar{\mathscr{F}}_{k,w}$ (note that we have taken the dual of the sheaf in the reference). By the comparison theorem in étale cohomology, those two cohomologies are canonically isomorphic, so we do not make any distinction unless otherwise specified. The projective limit $S = \varprojlim_K S_K$ admits a right $D^\times(\mathbb{A}^\infty)$-action by the multiplication from the right, and the E_λ-sheaf

$$\bar{\mathscr{F}}^D_{k,w,\ E_\lambda} = \bar{\mathscr{F}}^D_{k,w} \otimes_{\mathscr{O}_\lambda} E_\lambda$$

is $D^\times(\mathbb{A}^\infty)$-equivariant by the construction.

The lattice structure $\bar{\mathscr{F}}^D_{k,w}$is preserved by the $D^\times(\mathbb{A}^{\ell,\infty})$-action.

When g' is one, the sheaf $\bar{\mathscr{F}}^D_{k,w}$ is canonically defined over F by the theory of canonical models, which we denote by $\mathscr{F}^D_{k,w}$. By [4] 2.6, $\mathscr{F}^D_{k,w}$ is pure of weight w. This canonical F-structure gives a continuous G_F-action on the cohomology, giving the decomposition

$$H^1_{\text{ét}}(S_{K,\bar{F}},\ \bar{\mathscr{F}}_{k,w} \otimes E_\lambda) = \oplus_\pi \rho_{\pi,\lambda} \otimes_{\mathscr{O}_\lambda} (\pi^\infty)^{\tilde{K}}$$

for cuspidal representations π of $D^\times(\mathbb{A}_F)$ with infinity type (k,w) which does not factor through the reduced norm ([5], 2.2) assuming that E_λ is sufficiently large. By infinity type (k,w), we mean that π_∞ has the form

$$\pi_\infty = D_{k_{\iota_1},w} \otimes_{v \in I_F, v \neq \iota_1} \bar{D}_{k_v,w}$$

(see [5] §0 for the notation $\bar{D}_{k_v,w} = (\iota\det)^{-k'_\iota}\,\mathrm{Sym}^{\vee \otimes k_\iota - 2}$, which corresponds to $D_{k_v,w}$ by the Jacquet-Langlands correspondence [16]).

3.3. **Hecke algebras and correspondences.** We keep the same notations as in 3.2. We assume that K is locally factorizable, i.e., $K = \prod_u K_u$, and D is split at all $v|\ell$. For a finite set of finite places Σ containing $\{v;\ v|\ell\}$, let $H(D^\times(\mathbb{A}^{\Sigma,\infty}),\ K^\Sigma)$ be the convolution algebra over $\mathscr{O}_\lambda$. Namely, $H(D^\times(\mathbb{A}^{\Sigma,\infty}),\ K^\Sigma)$ is the set of the compactly supported $\mathscr{O}_\lambda$-valued smooth K^Σ-biinvariant functions on $D^\times(\mathbb{A}^{\Sigma,\infty})$. The algebra structure is given by the convolution.

We consider the action of $H(D^\times(\mathbb{A}^{\ell,\infty}),\ K^\ell)$ on the cohomology of $S_{\tilde{K}}$ induced from the adelic right action on $S = \varprojlim_{\tilde{K}} S_{\tilde{K}}$. We briefly review the basic facts since the relationship with the Verdier duality is subtle and important for our purpose.

For two compact open subgroups $K,\ K' \subset D^\times(\mathbb{A}^\infty)$, $g \in D^\times(\mathbb{A}^\infty)$ define an algebraic correspondence $[KgK']$: the first projection $S_{K \cap g^{-1}K'g} \to S_K$ and the second $S_{K \cap g^{-1}K'g} \to S_{g^{-1}K'g} \simeq S_{K'}$.

The correspondence induced by KgK' from S_K to $S_{K'}$ is dual to $K'g^{-1}K$ from $S_{K'}$ to S_K by the definition.

Since $\mathscr{F}_{k,w,E_\lambda}$ is $D^\times(\mathbb{A}^\infty)$-equivariant by the construction, $[KgK']$ gives

$$[KgK'] : R\Gamma(S_K,\ \bar{\mathscr{F}}_{k,w}) \otimes_{\mathscr{O}_\lambda} E_\lambda \to R\Gamma(S_{K'},\ \bar{\mathscr{F}}_{k,w}) \otimes_{\mathscr{O}_\lambda} E_\lambda$$

via cohomological correspondences they define.

We fix a uniformizer p_v of F_v at each finite place v. If the v-component g_v of $g \in D^\times(\mathbb{A}^\infty)$ satisfies

$$\mathrm{GL}_2(o_v) g_v \mathrm{GL}_2(o_v) \text{ is represented by } \begin{pmatrix} p_v^a & 0 \\ 0 & p_v^b \end{pmatrix} \text{ with } a,\ b \geq 0 \text{ for all } v|\ell$$

the $\mathscr{O}_\lambda$-lattice structure $\mathscr{F}_{k,w}$ is preserved, and

$$[KgK'] : R\Gamma(S_K,\ \bar{\mathscr{F}}_{k,w}) \to R\Gamma(S_{K'},\ \bar{\mathscr{F}}_{k,w})$$

is induced. We call this the *standard* action of $[KgK']$. Especially, we have the action of $H(D^\times(\mathbb{A}^{\Sigma,\infty}),\ K^\Sigma)$ for $K = K_\Sigma \cdot K^\Sigma$ by extending $K^\Sigma g K^\Sigma$ to $K\tilde{g}K$, $\tilde{g} = ((1_{D(F_v)})_{v \in \Sigma}, g)$, $g \in D^\times(\mathbb{A}^{\Sigma,\infty})$.

It is a natural question to ask whether there is a complex of $H(D^\times(\mathbb{A}^{\Sigma,\infty}),\ K^\Sigma)$ -modules which represents $R\Gamma(S_K, \bar{\mathscr{F}}_{k,w})$ or not. The answer is affirmative, which we state it in the form of proposition. In section 4, it becomes quite important.

Proposition 3.1. *There is a complex $L^\cdot$ of $H(D^\times(\mathbb{A}^{\Sigma,\infty}),\ K^\Sigma)$-modules bounded from below which represents $R\Gamma(S_K, \bar{\mathscr{F}}_{k,w})$ in $D^+(S_K, \mathscr{O}_\lambda)$. The induced action of $H(D^\times(\mathbb{A}^{\Sigma,\infty}),\ K^\Sigma)$ on $H^q(S_K, \bar{\mathscr{F}}_{k,w})$ coincides with the induced action by the Hecke correspondences.*

Proof. We work with the Betti version for simplicity (this case is sufficient for our later purpose). Let $L^{\cdot}$ be the Godement's canonical resolution of $\bar{\mathscr{F}}_{k,w}$. Since all the maps in defining the algebraic correspondences are finite and étale, cohomological operations defining the action of a cohomological correspondence are actually defined on $L^{\cdot}$: for example, for a finite étale morphism $f : X \to Y$, the trace map $f_! f^! \mathscr{F} = f_* f^* \mathscr{F} \to \mathscr{F}$ for a sheaf $\mathscr{F}$ is defined by

$$\sum_{y \in f^{-1}(x)} \mathscr{F}_y \to \mathscr{F}_x, \ (f_y)_{y \in f^{-1}(x)} \to \sum_{y \in f^{-1}(x)} f_y$$

on the level of stalk, which gives a lift of the trace map to Godement resolution $L^{\cdot}$. In this way one has an action of the convolution algebra on $\Gamma(L^{\cdot})$, and $R\Gamma(S_K, \ \mathscr{F}_{k,w})$ belongs to the derived category of $H(D^{\times}(\mathbb{A}^{\Sigma,\infty}), \ K)$-modules bounded below. Especially, for $K = \prod K_v$, the actions at two different places commute.

□

Remark 3.2. *For the étale cohomology with finite coefficients, the claim is proved by the same argument.*

3.4. **Duality formalism.** Take a finite set of finite places Σ containing $\{v; \ v|\ell\}$. Assume $\tilde{K} = \prod_u \tilde{K}_u$, and D is split at all finite $v \notin \Sigma$.

Note that

$$(\dagger) \qquad \bar{\mathscr{F}}_{k,w}^{\vee} \simeq \bar{\mathscr{F}}_{k,-w}.$$

giving a perfect pairing in the derived category of $\mathscr{O}_\lambda$-modules

$$R\Gamma(S_K, \ \bar{\mathscr{F}}_{k,w}) \otimes^{\mathbb{L}}_{\mathscr{O}_\lambda} R\Gamma(S_K, \ \bar{\mathscr{F}}_{k,-w}) \to \mathscr{O}_\lambda(-g')[-2g']$$

by the Poincaré duality. For the $D^{\times}(\mathbb{A}^{\Sigma,\infty})$-action, if we consider the standard action of g on $R\Gamma(S_K, \ \bar{\mathscr{F}}_{k,w})$, by the Poincaré duality, this corresponds to the standard action of g^{-1} on $R\Gamma(S_K, \ \bar{\mathscr{F}}_{k,-w})$ since the isomorphism (†) sends g-action to g^{-1}.

For the relation between $H(D^{\times}(\mathbb{A}^{\Sigma,\infty}), \ K)$-action and the Verdier duality, we have the following proposition by discussion of section 3.3.

Proposition 3.3. *The standard action $R\Gamma(S_K, \ \bar{\mathscr{F}}_{k,w}) \to R\Gamma(S_{K'}, \ \bar{\mathscr{F}}_{k,w})$ induced by $[KgK']$ is dual to the standard action $R\Gamma(S_{K'}, \ \bar{\mathscr{F}}_{k,-w})(g')[2g'] \to R\Gamma(S_K, \ \bar{\mathscr{F}}_{k,-w})(g')[2g']$ by $[K'g^{-1}K]$.*

We have two geometric actions of the convolution algebra, which we call the *standard* action and the *dual* action. By the dual action of $[KgK]$ on $R\Gamma(S_K, \bar{\mathscr{F}}_{k,w})$, we mean the standard action of $[Kg^{-1}K]$.

By proposition 3.3, the standard action of T_Σ becomes the dual action by the Poincaré duality.

We define standard Hecke operators. We choose a unformizer p_v of F_v for any finite place v. $a(p_v)$ (resp. $b(p_v)$) is the element in $D^{\times}(\mathbb{A}^{\infty})$ having $\begin{pmatrix} 1 & 0 \\ 0 & p_v \end{pmatrix}$ (resp. $\begin{pmatrix} p_v & 0 \\ 0 & p_v \end{pmatrix}$) as it's v-component, and 1 as other components.

Then the Hecke algebra $T_\Sigma = H(D^{\times}(\mathbb{A}^{\Sigma,\infty}), \ K^{\Sigma})$ over $\mathscr{O}_\lambda$ is isomorphic to the $\mathscr{O}_\lambda$-algebra generated by indeterminates $[T_v]$, $[T_{v,v}]$, $v \notin \Sigma$, adding $[T_{v,v}]^{-1}$ for $v \notin \Sigma$. Here $[T_v]$ is given by the characteristic function of $\tilde{K}^{\Sigma} a(p_v) \tilde{K}^{\Sigma}$, $[T_{v,v}]$ is given by the characteristic function of $\tilde{K}^{\Sigma} b(p_v) \tilde{K}^{\Sigma}$. For a T_Σ-module M, we define the dual T_Σ-action on M by

$$\begin{cases} [T_v] \mapsto [T_{v,v}]^{-1} \cdot [T_v] \\ [T_{v,v}] \mapsto [T_{v,v}]^{-1}. \end{cases}$$

Example 3.4. *For any continuous mod ℓ Galois representation $\bar{\rho} : G_\Sigma \to \mathrm{GL}_2(\bar{k}_\lambda)$, we define a maximal ideal $m_{\bar{\rho}}$ of T_Σ by*

$$[T_v] \mapsto \mathrm{trace}\bar{\rho}(\mathrm{Fr}_v),\ \ [T_{v,v}] \mapsto q_v^{-1}\mathrm{det}\bar{\rho}(\mathrm{Fr}_v).$$

The maximal ideal corresponding to the dual action is $m_{(\mathrm{det}\bar{\rho})^{-1}\otimes\bar{\rho}(1)}$.

In case of Shimura curves, the definition is compatible with the T_Σ-action on cohomology groups:

By choosing the canonical resolution, $R\Gamma(S_{\tilde{K}},\ \bar{\mathscr{F}}_{k,w})$ belongs to the derived category of T_Σ-complexes bounded below, sending $[T_v]$ and $[T_{v,v}]$ to the standard actions of $[\tilde{K}\begin{pmatrix} 1 & 0 \\ 0 & p_v \end{pmatrix}\tilde{K}]$ and $[\tilde{K}\begin{pmatrix} p_v & 0 \\ 0 & p_v \end{pmatrix}\tilde{K}]$. Then the dual action defined by the Poincaré duality is given by the above dual T_Σ-action since they are defined by $[\tilde{K}\begin{pmatrix} 1 & 0 \\ 0 & p_v^{-1} \end{pmatrix}\tilde{K}]$ and $[\tilde{K}\begin{pmatrix} p_v^{-1} & 0 \\ 0 & p_v^{-1} \end{pmatrix}\tilde{K}]$.

3.5. **Modules of type ω.**

We keep the notations, especially $T_\Sigma = H(D^\times(\mathbb{A}^{\Sigma,\infty}),\ K^\Sigma)$. We define a class of T_Σ-modules.

Definition 3.5. *Consider the category $\mathscr{C}_{T_\Sigma}$ of T_Σ-modules which are finitely generated as $\mathscr{O}_\lambda$-modules. We call an object N in $\mathscr{C}_{T_\Sigma}$ of type ω*

1. *if it has a finite length, then for any constituant N' appearing in the Jordan-Hölder sequence*
$$[T_v]^2 = [T_{v,v}](1+q_v)^2$$
holds on N' and for $v \notin \Sigma$,
2. *in general N is of type ω if and only if the graded modules $\lambda^n N/\lambda^{n+1}N$ ($n \in \mathbb{N}$) for the λ-adic filtration are all of type ω.*
3. *A maximal ideal m of T_Σ is of type ω if T_Σ/m is of type ω.*

By $\mathscr{C}_\Omega$ we denote the subcategory of $\mathscr{C}_{T_\Sigma}$ consisting of the T_Σ-modules of type ω.

It is easy to see that $\mathscr{C}_\Omega$ forms a Serre subcategory of $\mathscr{C}_{T_\Sigma}$, and is stable under the dual action of T_Σ. By $\mathscr{C}_{N\Omega}$, we mean the quotient category of $\mathscr{C}_{T_\Sigma}$ by $\mathscr{C}_\Omega$.

A typical example of modules of type ω is obtained by a one dimensional representation $\chi : D^\times(\mathbb{A}^{\Sigma,\infty}) \to E_\lambda^\times$. The induced T_Σ-action gives a module of type ω.

The following proposition shows that the maximal ideals of T_Σ of type ω correspond to very special reducible representations.

Proposition 3.6. *Assume that a continuous representation $\bar{\rho} : G_\Sigma \to \mathrm{GL}_2(\bar{k}_\lambda)$ satisfies*

$$(\dagger) \qquad \mathrm{trace}\bar{\rho}(\mathrm{Fr}_v)^2 = (1+q_v)^2 q_v^{-1}\mathrm{det}\bar{\rho}(\mathrm{Fr}_v)$$

for all $v \notin \Sigma$. Then $\bar{\rho}$ is reducible, and the semi-simplification $\bar{\rho}^{ß}$ satisfies

$$\bar{\rho}^{ß} \simeq \bar{\chi} \oplus \bar{\chi}(1)$$

for some one dimensional character $\bar{\chi} : G_\Sigma \to k_\lambda^\times$. In other words, the maximal ideal $m_{\bar{\rho}}$ corresponding to $\bar{\rho}$is of type ω if and only if $\bar{\rho}^{ß} \simeq \bar{\chi} \oplus \bar{\chi}(1)$ for some $\bar{\chi}$.

Proof. Let $\omega = \chi_{\mathrm{cycle}} \mod \ell$ be the Teichmüller character. By the Chebotarev density theorem, 3.6 ($\dagger$) is equivalent to

$$\mathrm{trace}(g; \mathrm{ad}^0\bar{\rho}) = \mathrm{trace}(g; 1 \oplus \omega \oplus \omega^{-1})$$

for any $g \in G_\Sigma$. Since $\mathrm{ad}^0\bar{\rho}$ satisfies $\Lambda^3\mathrm{ad}^0\bar{\rho} \simeq k$ and self-dual, the above equality of the traces implies the equality of characteristic polynomials

$$\det(1 - gT; \mathrm{ad}^0\bar{\rho}) = \det(1 - gT; 1 \oplus \omega \oplus \omega^{-1})$$

for any $g \in G_\Sigma$. By the Brauer-Nesbit theorem, the equality of semi-simplifications

$$(\mathrm{ad}^0\bar{\rho})^{ß} \simeq 1 \oplus \omega \oplus \omega^{-1}$$

as G_Σ-modules follows. If $\bar{\rho}$ is irreducible, the only possibility is to have ω or ω^{-1} as a submodule of $\mathrm{ad}^0\rho$, and hence $[F(\zeta_\ell) : F] = 2$ and ρ is induced from $F(\zeta_\ell)$. But this type of representations do not have the adjoint representation of the above form. We conclude that $\bar{\rho}$ is reducible, and the claim follows.

□

Remark 3.7. *The notion of modules of type ω is stronger than the notion of Eisenstein modules in* [7].

3.6. **Cohomological lemmas.** Let S be a strict trait with the generic point η and the closed point s. $I = \mathrm{Gal}(\bar{\eta}/\eta)$. Here $\bar{\eta}$ is a geometric point over η.

Lemma 3.8. *Let X be a flat regular scheme of finite type over S. Assume ℓ is prime to the residual characteristic of S, and take a coefficient ring Λ which is finite over $\mathbb{Z}_\ell$. For a Λ-smooth sheaf $\mathscr{F}$ on X, there are two exact sequences*

(a)
$$0 \to H^0_{\text{ét}}(X_{\bar{\eta}},\ \bar{\mathscr{F}})(-1)_I \to H^1_{\text{ét}}(X_\eta,\ \mathscr{F}_\eta) \to H^1_{\text{ét}}(X_{\bar{\eta}},\ \bar{\mathscr{F}})^I \to 0,$$

(b)
$$0 \to H^1_{\text{ét}}(X,\ \mathscr{F}) \to H^1_{\text{ét}}(X_\eta,\ \mathscr{F}_\eta) \to \prod_{Y \in J} H^0_{\text{ét}}(Y_{\mathrm{reg}},\ \mathscr{F}|_{Y_{\mathrm{reg}}})(-1).$$

Here J is the set of irreducible components of X_s, and $(-)_{\mathrm{reg}}$ means the regular locus of $(-)$.

Proof. The exactness of 3.12 (a) follows from the Hochshield-Serre spectral sequence applied to the morphism from $X_{\bar{\eta}}$ to X_η. We show 3.12 (b). $H^0_{\text{ét}}(Y_{\mathrm{reg}},\ \mathscr{F}|_{Y_{\mathrm{reg}}})(-1)$ is canonically isomorphic to $H^1_{\text{ét}}(\mathrm{Frac}\, o_{X,\bar{\eta}_Y},\ \mathscr{F}_{\bar{\eta}_Y})$, where $\bar{\eta}_Y$ is a geometric generic point of Y. Then the exactness of 3.12 (b) is equivalent to the following claim.

Claim 3.9. *An $\mathscr{F}_\eta$-torsor over X_η which becomes unramified at all maximal points of X_s extends to an $\mathscr{F}$-torsor over X uniquely up to isomorphism.*

3.9 is a consequence of the purity theorem of Zariski-Nagata [11], which says that, if X is regular, the category of the étale coverings of X is equivalent to the subcategory of the étale coverings of X_η which becomes unramified at all maximal points of X_s. □

Fix a commutative $\mathscr{O}_\lambda$-algebra A. We assume the following conditions on $(A, X, \mathscr{F})$.

Assumption 3.10. *For any $i \in \mathbb{Z}$,*

(1) *$H^i_R = H^i_{\text{ét}}(X_{\bar{\eta}},\ \bar{\mathscr{F}} \otimes_{\mathscr{O}_\lambda} R)$, $\mathscr{H}^i_R = H^i_{\text{ét}}(X,\ \mathscr{F} \otimes_{\mathscr{O}_\lambda} R)$ are (A, R)-bimodules for any commutative finite $\mathscr{O}_\lambda$-algebra R which commutes with I-actions.*

(2) *$H^i_R \to H^i_{R'}$ is an A-module homomorphism for any $\mathscr{O}_\lambda$-algebra homomorphism $R \to R'$.*

(3) *All homomorphisms in the long cohomology exact sequence $\{H^i_R\}_{i\in\mathbb{Z}}$ are A-module homomorphisms, functorial for any ring extension $R \to R'$.*

Lemma 3.11. *Fix a maximal ideal m of A. Let $\mathscr{E}$ be a Serre subcategory of the category of A-modules (A-mod), consisting of $\mathscr{O}_\lambda$-modules of finite length satisfying the following conditions.*

1. *$\mathscr{E}$ is stable under $(-)\otimes_{\mathscr{O}_\lambda} R$ for any finite $\mathscr{O}_\lambda$-algebra R.*
2. *The localization M_m at m of any element M in $\mathscr{E}$ vanishes.*
3. *X is a proper flat regular curve over S with the smooth generic fiber, and $\mathscr{F}$ is an $\mathscr{O}_\lambda$-smooth sheaf on X. Assumption 3.10 is satisfied for $(A, X, \mathscr{F})$.*
4. *$H^0_{\text{ét}}(X_{\bar\eta},\ \bar{\mathscr{F}}\otimes_{\mathscr{O}_\lambda} R)$, $H^2_{\text{ét}}(X_{\bar\eta},\ \bar{\mathscr{F}}\otimes_{\mathscr{O}_\lambda} R)\in\mathscr{E}$ for any $\mathscr{O}_\lambda$-module R of finite length.*
5. *$H^0_{\text{ét}}(X_s,\ \mathscr{F}|_{X_s}\otimes_{\mathscr{O}_\lambda} R)$, $H^2_c(Y_{\text{reg}},\ \mathscr{F}|_{Y_{\text{reg}}}\otimes_{\mathscr{O}_\lambda} R)\in\mathscr{E}$ for any irreducible component Y of X_s and for any R of finite length.*

Put $H_R = H^1_R$, $\mathscr{H}_R = \mathscr{H}^1_R$. Then the following hold:

(a) *$H_{R,m}$ is $\mathscr{O}_\lambda$-free, and $H_{R,m}\otimes_{\mathscr{O}_\lambda} k_\lambda \simeq H_{R/\lambda R,m}$ if R is $\mathscr{O}_\lambda$-flat.*
(b) *$\mathscr{H}_{R,m}$ is $\mathscr{O}_\lambda$-free, and $(\mathscr{H}_{R,m})\otimes_R k_\lambda \simeq \mathscr{H}_{R/\lambda R,m}$ if R is $\mathscr{O}_\lambda$-flat.*
(c) *$\mathscr{H}_{R,m}\simeq (H_{R,m})^I$ if R is $\mathscr{O}_\lambda$-flat.*

Proof. First note that M_m is finitely generated as an $\mathscr{O}_\lambda$-module for any A-module M, finite over $\mathscr{O}_\lambda$, since the A-action factors through a subalgebra in $\mathrm{End}_{\mathscr{O}_\lambda} M$, and hence a finite $\mathscr{O}_\lambda$-algebra, which is a product of local rings.

We prove (a). By the exact sequence

$$H^0_{\text{ét}}(X_{\bar\eta},\ \bar{\mathscr{F}}\otimes_{\mathscr{O}_\lambda} R/\lambda R)\to H^1_{\text{ét}}(X_{\bar\eta},\ \bar{\mathscr{F}}\otimes_{\mathscr{O}_\lambda} R)\xrightarrow{\lambda} H^1_{\text{ét}}(X_{\bar\eta},\ \bar{\mathscr{F}}\otimes_{\mathscr{O}_\lambda} R)$$

$H_{R,m}$ is an $\mathscr{O}_\lambda$-flat module since $H^0_{\text{ét}}(X_{\bar\eta},\ \bar{\mathscr{F}}\otimes_{\mathscr{O}_\lambda} R/\lambda R)$ vanishes after localization at m.

$$H^1_{\text{ét}}(X_{\bar\eta},\ \bar{\mathscr{F}}\otimes_{\mathscr{O}_\lambda} R)\xrightarrow{\lambda} H^1_{\text{ét}}(X_{\bar\eta},\ \bar{\mathscr{F}}\otimes_{\mathscr{O}_\lambda} R)\to H^1_{\text{ét}}(X_{\bar\eta},\ \bar{\mathscr{F}}\otimes_{\mathscr{O}_\lambda} R/\lambda R)\to H^2_{\text{ét}}(X_{\bar\eta},\ \bar{\mathscr{F}}\otimes_{\mathscr{O}_\lambda} R/\lambda R)$$

is exact, and the claim for H_R follows.

For (b), the $\mathscr{O}_\lambda$-freeness is proved similary as above once we know that $H^0_{\text{ét}}(X_s,\ \mathscr{F}|_{X_s}\otimes_{\mathscr{O}_\lambda} R)\in\mathscr{E}$. This follows from

$$H^0_{\text{ét}}(X_s, \mathscr{F}|_{X_s}\otimes_{\mathscr{O}_\lambda} R))\overset{(1)}{=} H^0_{\text{ét}}(X, \mathscr{F}\otimes_{\mathscr{O}_\lambda} R)\subset H^0_{\text{ét}}(X_\eta, \mathscr{F}_\eta\otimes_{\mathscr{O}_\lambda} R)$$

and by (2). Here (1) is an isomorphism by the proper base change theorem. For the rest of (b), it suffices to see $H^2_{\text{ét}}(X_s,\ \mathscr{F}|_{X_s}\otimes_{\mathscr{O}_\lambda} R/\lambda R)\in\mathscr{E}$. This follows from

$$H^2_{\text{ét}}(X_s,\ \mathscr{F}|_{X_s}\otimes_{\mathscr{O}_\lambda} R/\lambda R) = \oplus_{Y\in J} H^2_c(Y_{\text{reg}},\ \mathscr{F}|_{Y_{\text{reg}}}\otimes_{\mathscr{O}_\lambda} R/\lambda R).$$

For (c), consider the commutative diagram

$$\begin{tikzcd}
\mathscr{H}_{R,m}\otimes_{\mathscr{O}_\lambda} k_\lambda \arrow[r, "(1)"] \arrow[d, "(2)"'] & H^I_{R,m}\otimes_{\mathscr{O}_\lambda} k_\lambda \arrow[d, "(4)"] \\
\mathscr{H}_{R/\lambda R,m} \arrow[r, "(3)"] & H^I_{R/\lambda R,m}.
\end{tikzcd}$$

(2) is an isomorphism by (b). (3) is an isomorphism by lemma 3.12. The composition

$$H^I_{R,m}\otimes_{\mathscr{O}_\lambda} k_\lambda \xrightarrow{(4)} H^I_{R/\lambda R,m}\subset H_{R/\lambda R,m} = H_{R,m}\otimes_{\mathscr{O}_\lambda} k_\lambda$$

is injective since $H^I_{R,m}\subset H_{R,m}$ is an $\mathscr{O}_\lambda$-direct summand by the definition and (a). So the map (1) is an isomorphism. $\mathscr{H}_{R,m}\simeq H^I_{R,m}$ follows since both modules are $\mathscr{O}_\lambda$-finite free. $\square$

Remark 3.12. *Since $\mathscr{C}$ is a Serre subcategory, it suffices to check the conditions for $R = A/m$.*

4. Proof of theorem B in the ramified case

4.1. **Auxiliary places.** Technically, we need to choose some auxiliary place y such that the discrete subgroups are torsion-free, and the Hecke algebra does not introduce essentially new component at y, the idea introduced by F. Diamond and R. Taylor ([8], lemma 11). The extra assumption on $\bar{\rho}$ when $[F(\zeta_\ell):F]=2$ is necessary to make this change possible (this condition seems to be natural since $\mathrm{PGL}_2(F)$ has a non-trivial ℓ-torsion elements in this case).

Lemma 4.1. *[existence of an auxiliary place] Assume that $\ell \neq 2$ and $\bar{\rho}$ is absolutely irreducible. Assume moreover that $\bar{\rho}|_{F(\zeta_\ell)}$ is absolutely irreducible if $[F(\zeta_\ell):F]=2$. Then there are infinitely many finite places y such that $q_y \not\equiv 1(\mod \ell)$ and for the eigenvalues $\bar{\alpha}_y$, $\bar{\beta}_y$ of $\bar{\rho}(\mathrm{Fr}_y)$, $\bar{\alpha}_y \neq q_y^{\pm 1}\bar{\beta}_y$ holds.*

As in [8], this follows from the Chebotarev density theorem and the following lemma 4.2, which we give a proof since the linear disjointness of F and $\mathbb{Q}(\zeta_\ell)$ is assumed in the reference. The proof given here does not use the classification of subgroups of $\mathrm{GL}_2(\mathbb{F}_{\ell^n})$.

In the following, we only consider representations defined over a field of characteristic different from 2. We say an absolutely irreducible two dimensional representation ρ is monomial if it induced from an index 2 subgroup of G. An absolutely irreducible representation ρ is monomial if the restriction $\rho|_H$ to a normal subgroup H of G is a sum of two distinct characters. Equivalently, an absolutely irreducible representation ρ is monomial if $\mathrm{ad}^0\rho$ is absolutely reducible.

Lemma 4.2. *Let k be a field of characteristic $\ell \neq 2$, G be a finite group, $\rho : G \to \mathrm{GL}_2(k)$ be an absolutely irreducible representation, and $\chi : G \to k^\times$ be an even order character. Assume that*

(∗) *For any $g \in G$ with $\chi(g) \neq 1$, $\rho(g)$ has eigenvalues of the form $\{\alpha,\ \chi(g)\alpha\}$.*

Then χ has order 2, and ρ is induced from a character of H. Here $H = \ker\chi$.

Proof of lemma 4.2. We enlarge k so that all eigenvalues of $\rho(g)$, $g \in G$, belong to k. Put $Z = \{g \in G,\ \rho(g) \text{ acts as a scalar}\}$. By Schur's lemma, Z is the center of G. Z is a normal subgroup of $H = \ker\chi$, since χ factors through $G \to G/Z \to k^\times$ by (∗). $H \neq Z$ since χ is a character of an even order.

We need the following sublemma.

Sublemma 4.3. *Assumptions are as in 4.2. Then the following hold:*

(a) *If $\chi(g)$ has order $d > 1$, $g^d \in Z$.*
(b) *H/Z is an abelian group.*
(c) *Let $G' = \chi^{-1}(\{\pm 1\})$. Then G'/Z is also abelian if H/Z is of type $(2,\dots,2)$.*

Proof of 4.3 . For (a), note that $\rho(g)$ is semi-simple, and $\rho(g)^d$ is a scalar matrix. For (b), take an element $c \in G'$ such that $\chi(c) = -1$. For any $h \in H$, $(c\cdot h)^2 \in Z$, especially $c^2 \in Z$, by (a). This means that in H/Z, $chc^{-1} = h^{-1}$ holds, and hence H/Z is abelian because the map $h \mod Z \mapsto h^{-1} \mod Z$ is a group homomorphism. G' is a semi-direct product of $\mathbb{Z}/2$ by H/Z. (c) is clear since the adjoint action of c is trivial on H/Z by the assumption. □

We return to the proof of lemma 4.2. We show $\rho|_H$ is reducible. Assume contrary. Then the adjoint representation $\mathrm{ad}^0\rho|_H$ is reducible, and the irreducible consitituants are one dimensional characters since it factors through an abelian group H/Z. Thus $\rho|_H$ is monomial. H/Z is isomorphic to $\mathbb{Z}/2$ or $\mathbb{Z}/2 \times \mathbb{Z}/2$ since it is dihedral and abelian. Since H/Z is of type $(2,\dots,2)$, G'/Z is also abelian by 4.3 (c). By the same argument, G'/Z is

$\mathbb{Z}/2$ or $\mathbb{Z}/2\times\mathbb{Z}/2$, and hence G'/Z is $\mathbb{Z}/2\times\mathbb{Z}/2$, and H/Z is $\mathbb{Z}/2$. In this case, it is easy to see that $\mathrm{ad}^0\rho|_{G'}$ is a sum of three different non-trivial characters of G'/Z, and hence $\rho|_H$ can not be irreducible, which leads to a contradiction.

Next we show that $\rho|_H$ is a sum of two distinct characters. If it is indecomposable, the unique one dimensinal subrepresentation of H is fixed by the action of G, and ρ is reducible. So $\rho|_H$ is decomposable, say $\chi_1\oplus\chi_2$. We show $\chi_1\neq\chi_2$. If not, $\mathrm{ad}^0\rho|_H\simeq 1^{\oplus 3}$. $(*)$ implies

$$\mathrm{trace}(g;\mathrm{ad}^0\rho)=\mathrm{trace}(g,1\oplus\chi\oplus\chi^{-1})$$

for $g\in G\setminus H$, and hence $(*)$ holds for any $g\in G$. ρ is reducible by the argument of 3.6, which leads to a contradiction.

We finish the proof of 4.2. Since $\rho|_H$ is a sum of two distinct characters, ρ is monomial. G/Z is dihedral, and the image of χ is a quotient of a dihedral group. This implies that χ has order 2, and H is an index 2 subgroup of G. ρ is induced from a character of H. $\square$

4.2. **Proof of theorem B.** We prove theorem B in the introduction.

Theorem 4.4. *[Theorem B] Let $\bar\rho: G_F\to\mathrm{GL}_2(\bar k_\lambda)$ be a continuous absolutely irreducible mod ℓ-representation satisfying A1)-A4):*

- A-1) *$\ell\geq 3$, and $\bar\rho|_{F(\zeta_\ell)}$ is absolutely irreducible if $[F(\zeta_\ell):F]=2$.*
- A-2) *$\bar\rho\simeq\rho_{\pi,\lambda}\mod\lambda$, π a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of infinity type (k,w) defined over E_λ, satisfying $(\pi^\infty)^K\neq\{0\}$ for some compact open subgroup $K=\prod_u K_u$.*
- A-3) *for a place $v\not|\ell$, $\bar\rho$ is either ramified at v, or $q_v\equiv 1\mod\ell$.*
- A-4) *If g is even, assume that π_u is essentially square integrable for some $u\neq v$, $v\not|\ell$, with $K_u=K_1(m_u^{\mathrm{cond}\pi_u})$.*

Then there is a cuspidal representation π', having the same infinity type (k,w) as π, defined over a finite extension $E'_{\lambda'}\supset E_\lambda$ such that the following conditions hold:

1. *The associated λ'-adic representation $\rho_{\pi',\lambda'}$ gives $\bar\rho$. $({\pi'}^{v,\infty})^{K^v}\neq\{0\}$,*
2. *The conductor $\mathrm{cond}(\pi'_v)$ of π'_v is equal to $\mathrm{Art}\bar\rho|_{G_v}$, where G_v is the decomposition group at v, and $\mathrm{Art}\in\mathbb{Z}$ means the Artin conductor,*
3. *$\det\rho_{\pi',\lambda'}\cdot\chi_{\mathrm{cycle}}^{w+1}$ is the Teichmüller lift of $\det\bar\rho\cdot\bar\chi_{\mathrm{cycle}}^{w+1}$.*

Proof. We prove (3) in the next subsection (proposition 4.11). Under the assumption of theorem B, $\bar\rho=\rho_{\pi,\lambda}\mod\lambda$ for a cuspidal representation $\pi=\otimes_u\pi_u$ with $(\pi^\infty)^K\neq\{0\}$ for a compact open subgroup $K=K_v\cdot K^v$ of $\mathrm{GL}_2(\mathbb{A}_F^\infty)$. Take an integer $n\geq 0$ so that $K'=K(m_v^n)\cdot K^v\subset K$. $\bar\rho^{I_v}\neq\{0\}$. We seek for a cuspidal representation π' with $\mathscr{O}_{\lambda'}$-coefficient satisfying $({\pi'}^\infty)^{K'}\neq\{0\}$, and $\rho_{\pi',\lambda'}^{I_v}\neq\{0\}$.

We make use of a Shimura curve. Fix $\iota_1\in I_F$. Choose a division quaternion algebra D which is unramified outside ι_1 when g is odd, unramified outside u and ι_1 when g is even. $S_{\tilde K}$ denotes the canonical model over F of the Shimura curve associated to D and a compact open subgroup $\tilde K$ of $D^\times(\mathbb{A}^\infty)$. Our choice of $\tilde K$ is as follows.

By a Chebotarev density argument using lemma 4.1 and [12] lemma 7.1, we can take an auxiliary place y so that $g^{-1}K_{11}(y)g\cap\mathrm{SL}(D)_+,\ g\in D^\times(\mathbb{A}^\infty)$, are torsion free, $q_y\not\equiv 1\mod\ell$, and $\bar\alpha_y\neq q_y^{\pm1}\bar\beta_y$. Then we put

$$\tilde K=\begin{cases}K'\cap K_{11}(y) & (g:\ \text{odd})\\ \tilde K=(\tilde K_u\cdot {K'}^u)\cap K_{11}(y) & (g:\ \text{even}).\end{cases}$$

Here we fix a maximal order o_{D_u} of D_u, and $\tilde K_u=1+\Pi_u^{\mathrm{cond}\pi_u}\cdot o_{D_u}$where Π_u is a uniformizer of o_{D_u}.

We take a finite set of finite places Σ containing $\{v, v|\ell\}$, $\{v, v|\mathrm{cond}\pi\}$, y and u when g is even. We put $T_\Sigma = H(D^\times(\mathbb{A}^{\Sigma,\infty}), \tilde{K})$.

By replacing E_λ by a bigger ℓ-adic field E'_λ, $\mathscr{F}_{k,w}$ defined in 3.2, we have the decomposition

$$H^1_{\text{ét}}(S_{\tilde{K},\bar{F}},\ \bar{\mathscr{F}}_{k,w} \otimes_{\mathscr{O}_\lambda} E'_\lambda) = \oplus_\pi \rho_{\pi,\lambda} \otimes_{\mathscr{O}_\lambda} (\pi^\infty)^{\tilde{K}}.$$

Here we identify the index set as a set of cuspidal representations of $\mathrm{GL}_2(\mathbb{A}_F)$ (not of $D^\times(\mathbb{A})$) by the Jacquet-Langlands correspondence. By our assumption 4.4 A-2) and 4.4 A-4), we may assume that $\rho_{\pi,\lambda}$ appears in the decomposition of $H^1_{\text{ét}}(S_{\tilde{K},\bar{F}},\ \bar{\mathscr{F}}_{k,w})$.

By X we mean the arithmetic model of $S_{\tilde{K}}$ over o_v defined by Carayol [4] using Drinfeld basis, $\mathscr{F} = \mathscr{F}_{k,w}$. $I = I_v$ is the inertia group at v.

Note that the arithmetic model X is available for our $\tilde{K}$: In [4], Carayol assumes that the compact open subgroup defining a Shimura curve is sufficiently small. Note that this condition of smallness is satisfied if we replace $\tilde{K}$ by a smaller subgroup U with $U_v = \tilde{K}_v$, so it is also true for our $\tilde{K}$ since $S_{\tilde{K}}$ is obtained by a quotient of S_U by a finite group which acts freely by our choice of auxiliary place y. See also the remark on page 62 of [14].

For a finite $\mathscr{O}_\lambda$-algebra R, we put

$$H_R = H^1_{\text{ét}}(X_{\bar{F}_v},\ \bar{\mathscr{F}} \otimes_{\mathscr{O}_\lambda} R).$$

Note that H_R carries a natural $\mathscr{O}_\lambda[I] \times T_\Sigma$-module structure given by the standard action of Hecke operators.

Lemma 4.5. *For a maximal ideal m of T which is not of type ω, $H_{\mathscr{O}_\lambda,m}$ is $\mathscr{O}_\lambda$-free, and*

$$H^I_{\mathscr{O}_\lambda,m} \otimes_{\mathscr{O}_\lambda} k_\lambda = H^I_{k_\lambda,m}$$

holds.

Proof of lemma 4.5. We check the conditions of lemma 3.11, by taking $\mathscr{E}$ to be the category $\mathscr{C}_\Omega$ of modules of type ω.

Sublemma 4.6. *The T_Σ-actions on $H^0_{\text{ét}}(X_{\bar{F}_v},\ \bar{\mathscr{F}} \otimes_{\mathscr{O}_\lambda} k_\lambda)$ and $H^0_{\text{ét}}(X_{\bar{F}_v},\ \bar{\mathscr{F}}^\vee \otimes_{\mathscr{O}_\lambda} k_\lambda)$ are both of type ω.*

Proof of sublemma 4.6. The sheaf $\mathscr{F} \otimes_{\mathscr{O}_\lambda} k_\lambda$ is trivialized in a $D^\times(\mathbb{A}^{\Sigma,\infty})$-equivariant way by a finite covering $X'_{F_v} \to X_{F_v}$ corresponding to $\tilde{K}' = \prod_{v|\ell} K(m_v) \cdot \tilde{K}^\ell$ by the definition 3.2. Over X'_{F_v}, $D^\times(\mathbb{A}^{\Sigma,\infty})$-action induces the action of the convolution algebra of type ω, since it is obtained, on any constituants, from one dimensional actions of $D^\times(\mathbb{A}^{\Sigma,\infty})$ by

$$\pi_0(X_{\bar{F}_v}) \simeq \pi_0(F^\times \backslash \mathbb{A}^{\infty}{}^\times_F / \det \tilde{K}').$$

So the claim follows for H^0. Since the standard action on $H^2(X_{\bar{F}_v},\ \bar{\mathscr{F}} \otimes_{\mathscr{O}_\lambda} k_\lambda)$ is obtained from the dual action on $H^0(X_{\bar{F}_v},\ \bar{\mathscr{F}}_{k,-w} \otimes_{\mathscr{O}_\lambda} k_\lambda)$ by the Poincaré duality, by the same argument the claim also follows for H^2. □

By the proof of lemma 3.11, we have that $H_{\mathscr{O}_\lambda}$ is $\mathscr{O}_\lambda$-free ignoring modules of type ω. $H_{\mathscr{O}_\lambda} \otimes_{\mathscr{O}_\lambda} k_\lambda = H_{k_\lambda}$ in $\mathscr{C}_{N\Omega}$ since the T_Σ-actions on $H^0_{\text{ét}}(X_{\bar{F}_v},\ \bar{\mathscr{F}} \otimes_{\mathscr{O}_\lambda} k_\lambda)$ and $H^0_{\text{ét}}(X_{\bar{F}_v},\ \bar{\mathscr{F}}^\vee \otimes_{\mathscr{O}_\lambda} k_\lambda)$ are of type ω.

By [4] 9.4.3, the arithmetic model $X = X_{K(v^n)_v \cdot K^v}$ over $S = \operatorname{Spec} o_v$ of $S_{K(v^n)_v \cdot K^v}$ is regular, proper and flat over S, if K^v is sufficiently small. By the same holds for our $\tilde{K}$.

Here the set of irreducible components J_{K^v} of the geometric special fiber $X_{\bar{s}}$ is isomorphic to $\oplus_{L \in \mathbb{P}^1(o_v/v^n)} Y_{L,K^v}$, Y_{L,K^v} is a smooth curve, and there is a $D^\times(\mathbb{A}^{v,\infty})$-equivariant isomorphism

$$\pi_0(Y_{L,K^v}) \simeq \pi_0(F^\times \backslash \mathbb{A}_F^{\infty\times} / o_v^\times \cdot \det K^v).$$

From this description, it follows that conditions of b), c) of lemma 3.11 are satisfied for $\mathscr{F} = \mathscr{F}_{k,w}$, since $\mathscr{F}/\lambda^n\mathscr{F}$ is trivialized in an equivariant way by a finite covering $Y_{L,K'v} \to Y_{L,K}$, by the same argument as above.

So lemma 3.11 is applied, and lemma 4.5 is shown.

□

We return to the proof of theorem 4.4. By lemma 4.5, $\bar{\rho}$ appears in $H^I_{\mathscr{O}_\lambda,m} \otimes_{\mathscr{O}_\lambda} k_\lambda$, and for the corresponding maximal ideal m, which is not of type ω, T_m-module $H^I_{\mathscr{O}_\lambda,m} = H^I_{\mathscr{O}_\lambda} \otimes_T T_m$, $H^I_{\mathscr{O}_\lambda}$ localized at m, is $\mathscr{O}_\lambda$-free and non-zero, implying that there is a cuspidal representation π' with coefficient in $\mathscr{O}'_{\lambda'}$ such that $({\pi'}^\infty)^{K'} \neq \{0\}$, $\rho_{\pi',\lambda'}$ gives $\bar{\rho}$ and $\rho^{I_v}_{\pi',\lambda'} \neq \{0\}$.

The y-component of π' is spherical: since π'_y has a non-zero fixed vector by $K_{11}(m_y)$, π'_y belongs to principal series or (twisted) special representation. By our condition on y, the latter case does not occur since Fr_y-eigenvalues satisfies $\bar{\alpha}_y \neq q_y^{\pm 1}\bar{\beta}_y$. By condition $q_y \not\equiv 1 \mod \ell$, π'_y is spherical.

By [5], théorème A, the Artin conductor of $\rho'_v = \rho_{\pi',\lambda}|_{G_v}$ and the conductor of π'_v is the same for $v \nmid \ell$. By the formula for Artin conductors,

$$\mathrm{Art}\rho'_v = 2 - \dim_{E'_{\lambda'}} {\rho'_v}^{I_v} + \mathrm{sw}\rho'_v$$

$$\mathrm{Art}\bar{\rho}|_{G_v} = 2 - \dim_{k_{\lambda'}} \bar{\rho}^{I_v} + \mathrm{sw}\bar{\rho}|_{G_v}$$

hold. Here sw means the swan conductor. Since the swan conductor does not change under mod ℓ-reduction, $\mathrm{Art}\rho'_v = \mathrm{Art}\bar{\rho}|_{G_v}$ if and only if $\dim_{E'_{\lambda'}} {\rho'_v}^{I_v} = \dim_{k_{\lambda'}} \bar{\rho}^{I_v}$. We conclude that π' satisfies the desired equality

$$\mathrm{cond}(\pi'_v) = \mathrm{Art}\bar{\rho}|_{G_v}$$

since $\bar{\rho}$ is ramified at v. So ${\pi'}^\infty$ has a non-zero $K_1(m_v^{\mathrm{Art}\bar{\rho}|_{G_v}}) \cdot \tilde{K}^v$-fixed vector in the ramified cases. □

We have proved theorem B, except for the claim on the determinant.

Remark 4.7.
a) The equality $\mathscr{H}_R = H^I_R$ is seen as a local invariant cycle theorem for characteristic ℓ coefficient. We are mimicking the proof in the $\mathbb{Q}_\ell$-case, using Carayol's result for $K(v^n)$ that the corresponding arithmetic model is regular, plus the determination of the adéle action on the set of irreducible components of the special fiber. The regularity assures a purity property, in our case the Zariski-Nagata's purity theorem for étale coverings suffices.
b) Even in the unramified case, we get a cuspidal representation π' whose v-component π'_v has conductor at most one. By the determinant normalization in 4.3, we may replace π' again, and obtain π' with ${\pi'_v}^{K_0(m_v)} \neq \{0\}$. In this case we analyze a filtration on $\mathscr{H}_R$, and show the Mazur's principle in 5.1.
c) We may add U_w-operators for $w|\ell$ (or their modification) to T_Σ. By the same method, one can get a nearly ordinary π' starting from nearly ordinary π.

4.3. **Perfect complex argument.**

Lemma 4.8. *Let A be a noetherian local ring with maximal ideal m_A and the residue field k_A, B be an A-algebra. Let L be a complex of B-modules bounded below, having finitely generated cohomologies $H^i(L)$ as A-modules. For a maximal ideal m of B above m_A, assume $H^i(L \otimes^{\mathbb{L}}_A k_A) \otimes_B B_m$, the localization at m, is zero for $i \neq 0$. Then $H^0(L) \otimes_B B_m$ is A-free.*

Proof. Since B_m is B-flat, we may assume $B = B_m$ (finite generation assumption is satisfied after localization since m is above m_A). Then $H^i(L \otimes^{\mathbb{L}}_A k_A)$ is zero except $i = 0$. By taking the minimal free resolution as A-complexes, the claim follows. □

Lemma 4.9. *Let $\pi : X \to Y$ be an étale Galois covering with Galois group G. Let $\mathscr{F}$ be a smooth Λ-sheaf on Y. Then $R\Gamma(X, \pi^*\mathscr{F})$ is a perfect complex of $\Lambda[G]$-modules, and*

$$R\Gamma(X, \pi^*\mathscr{F}) \otimes^{\mathbb{L}}_{\Lambda[G]} \Lambda[G]/I_G \simeq R\Gamma(Y, \mathscr{F})$$

holds. Here I_G is the augmentation ideal, and the map is induced by the trace map.

This is known (especially in the dual form) in any standard cohomology theory.

Remark 4.10. *The above canonical morphism $R\Gamma(X, \pi^*\mathscr{F}) \to R\Gamma(Y, \mathscr{F})$ in $D^b_c(\Lambda)$ obtained by forgetting the G-action is given by the trace map.*

In case of Shimura curves, which is our main application of lemma 4.9 , the above morphism is compatible with the standard action of Hecke operators: it suffices to see the dual morphism $R\Gamma(Y,\ \mathscr{F}^\vee) \to R\Gamma(X,\ \mathscr{F}^\vee)$ is compatible with dual action of $[KgK]$. But this is the standard action of $[Kg^{-1}K]$ by proposition 3.3, and for the standard action, the compatibility is clear.

We apply the lemma to adjust the determinant, which was proved by Carayol [6] in case of $\mathbb{Q}$ (see [15] for the generalization).

Proposition 4.11. *[determinant optimization] Let π be a cuspidal representation of infinite type (k, w), having a non-zero fixed vector under $K_1(m_v^n) \cdot K^v$. Then there is π', π'^{∞} has a non-zero fixed vector under $(K_1(m_v^n) \cdot H) \cdot K^v$. Here*

$$H = \{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},\ \alpha \mod m_v^n \text{ has an } \ell\text{-power order}\}.$$

Proof. We take an auxiliary place y by lemma 4.1, and replace K by $K \cap K_{11}(y)$. Put $X = S_{K_1(v^n)_v \cdot K^v}$, $Y = S_{K_H(v^n)_v \cdot K^v}$, $\mathscr{F} = \bar{\mathscr{F}}_{k,w}$. We view X and Y as complex varieties. $\pi : X \to Y$ is an étale Galois covering with group $G = H/H'$, H' is the inverse image of a subgroup of $(o_v/m_v^n)^\times$ represented by units determined by K^v. We consider the complex L associated to Godement's canonical resolution of $\pi^*\mathscr{F}$ on X, which calculates (usual, not étale) sheaf cohomology $H^i(X,\ \pi^*\mathscr{F})$, and represents $R\Gamma(X,\ \pi^*\mathscr{F})$ in the derived category. Put $A = \mathscr{O}_\lambda[G]$, $B = A[[T_u],\ [T_{u,u}],\ [T_{u,u}]^{-1},\ u \notin \Sigma]$. The complex L admits a commuting action of Hecke operators, and of G by the discussion in 3.3, so we view L as a complex of B-modules, sending $[T_u]$, $[T_{u,u}]$ to the corresponding standard Hecke action at u. Then it follows that $H^i(S_{K_1(v^n)_v \cdot K^v}, \bar{\mathscr{F}}_{w,k})_m$ is $\mathscr{O}_\lambda[G]$-free by the previous lemmas ($i = 0$ or 1 according to g is even or not), since $H^0(Y,\ \mathscr{F} \otimes_{\mathscr{O}_\lambda} k_\lambda)$ and $H^2(Y,\ \mathscr{F} \otimes_{\mathscr{O}_\lambda} k_\lambda)$ are of type ω when g is odd as in 4.6. The G-invariant part is non-zero by the freeness. □

5. Mazur principle

In this section, we prove the following corollary A' in the introduction:

Claim 5.1. *[Corollary A' (Mazur principle)] Assumptions are as in theorem A. Then there exists π' as in theorem A when $\bar{\rho}$ is unramified at v, and $q_v \not\equiv 1 \mod \ell$.*

5.1. **The odd degree case.** In this subsection, we discuss 5.1 under condition A-1)-A-4) proved by Jarvis in [14]. We include this as a toy model for the Mazur principle in the even degree case in 5.3. We assume that $q_v \not\equiv 1 \mod \ell$. We may also assume that π_v is an unramified special representation by remark 4.7 b).

We return to the general setting as in 3.6. In addition to the assumption of lemma 3.11, we make the following additional assumption further:

Assumption 5.2. (1) *Any irreducible component Y of X_s is smooth.*
(2) *$\mathscr{F}_Y$ is pure of weight w for some integer w independent of Y.*

We define and analyze a standard filtration $W_R \subset \mathscr{H}_R = H^1_{\text{ét}}(X_s,\ \mathscr{F}_{X_s} \otimes_{\mathscr{O}_\lambda} R)$. Let J be the set of irreducible component of X_s, and Z be the set of singular points on $(X_s)_{\text{red}}$.

We define a skyscraper sheaf $\mathscr{G}_R$ supported on Z by

$$0 \to \mathscr{F} \otimes_{\mathscr{O}_\lambda} R \to \oplus_{Y \in J} i_Y(\mathscr{F}|_Y) \otimes_{\mathscr{O}_\lambda} R \to \mathscr{G}_R \to 0.$$

Then $W_R \subset \mathscr{H}_R$ is defined by

$$0 \to H^0_{\text{ét}}(X_s,\ \mathscr{F}_{X_s} \otimes_{\mathscr{O}_\lambda} R) \to \oplus_{Y \in J} H^0_{\text{ét}}(Y,\ \mathscr{F}_Y \otimes_{\mathscr{O}_\lambda} R) \to H^0_{\text{ét}}(Z,\ \mathscr{G}_R) \to W_R \to 0.$$

$$\mathscr{H}_R/W_R = \oplus_{Y \in J} H^1(Y,\ \mathscr{F}|_Y \otimes_{\mathscr{O}_\lambda} R)$$

follows from the definition. In $\mathscr{C}_{N\Omega}$, we have $W_R = H^0(Z,\ \mathscr{G}_R)$. Since $\mathscr{G}_{\mathscr{O}_\lambda}$ is $\mathscr{O}_\lambda$-smooth, it follows that the formation of W_R commutes with base change after localization at maximal ideal m, and we conclude that $\mathscr{H}_R/W_R$ has the same property.

We apply the formalism in 3.6 to Shimura curves $S_{D,\tilde{K}}$ and $\mathscr{F}_{k,w}$ with additional assumption 5.2. As in 4.5, we take $\mathscr{E}$ in lemma 3.11 to be the category of ω-type modules.

We choose D as in the proof of theorem 4.4. Our choice of compact open subgroup $\tilde{K} = \tilde{K}^v \cdot \tilde{K}_v$ is $\tilde{K}^v = K^v$, and for $\tilde{K}_v$

$$\tilde{K}_v = K(m_v^N) \cdot A, \qquad A = \{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},\ \alpha \in o_v^\times,\ \alpha \equiv 1 \mod m_v^N\},\ N \geq 1.$$

The assumption 5.2 (1) is satisfied for $K_v = K(m_v^N)$ by [4] 9.4.3, and the action of $A/A \cap K(m_v^N)$ is étale on the arithmetic model, so it is also satisfied with our $\tilde{K}$. The assumption 5.2 (2) on the weight of $\mathscr{F}$ is satisfied with weight w, since over a quadratic extension of F unramified at v, $\mathscr{F}$ is a subquotient of a (possibly higher) cohomology sheaf of an abelian scheme. This follows from [4], 2.6. See also [20] for the discussion.

Let T be the Hecke algebra generated by $[T_u],\ [T_{u,u}],\ [T_{u,u}]^{-1}$ for $u \notin \Sigma$ over $\mathscr{O}_\lambda$, m be the maximal ideal of T corresponding to $\bar{\rho}$.

Let P be the p-Sylow subgroup of $K_v/K(m_v^N)$, where p is the residue characteristic of v. It contains $K_{11}(m_v)/K(m_v^N)$. Since p is different from ℓ, the operation of taking the P-invariants of $\mathscr{H}_R$ commutes with any scalar extension. We define $U(p_v)$-operator for a fixed uniformizer p_v at v acting on $\mathscr{H}_R^P$ in the usual way: it is defined by double coset $K_0(m_v)a(p_v)K_0(m_v) \cdot K^v$ (see section 2 for the notation). $U(p_v)$-operator thus defined commutes with G_F-action since all group actions are defined over F.

(By the vanishing of $H^*(P,\ -)$,

$$H_R^P = H^1_{\text{ét}}(S_{K'},\ \mathscr{F}_{(k,w)} \otimes_{\mathscr{O}_\lambda} R)_m,\ K' = K^v \cdot (K_{11}(m_v)A),$$

and the $U(p_v)$-operator is equal to the one defined geometrically.)

By our assumption, $\mathscr{H}^P_{E_\lambda} \neq \{0\}$. Note that $(H^P_{k_\lambda})_m/m(H^P_{k_\lambda})_m$ as a Galois module is isomorphic to $\bar{\rho}^{\oplus\alpha}$ for some $\alpha \geq 1$ by the Eichler-Shimura relation [4], 10.3 and by the Boston-Lenstra-Ribet theorem [3] using the irreducibility of $\bar{\rho}$. As a T-module, it is isomorphic to $(T/m)^{2\alpha}$.

Assume the Hecke module T/m corresponding to $\bar{\rho}$ occurs in $\mathscr{H}^P_{k_\lambda}/W^P_{k_\lambda}$. Then it occurs in $(\mathscr{H}^P_{\mathscr{O}_\lambda}/W^P_{\mathscr{O}_\lambda})_m \neq \{0\}$, which is $\mathscr{O}_\lambda$-free. Let π be the corresponding cuspidal representation. Since $\mathscr{F}_Y$ is pure of weight w for each irreducible component Y of X_s, the weight here is $w+1$ by the Weil conjecture. If we assume that π_v is special, $\det\rho_{\pi,\lambda}(\mathrm{Fr}_v) = q_v\beta^2$, with $|\beta| = q_v^{\frac{w+1}{2}}$, which is impossible by the weight reason. It follows that π_v must be a principal series. Moreover, $\rho_v = \rho_{\pi,\lambda'}|_{G_v}$, associated to π_v by the local Langlands correspondence,

has a non-zero I_v-fixed part and $\det\rho_v|_{I_v}$ is the Teichmüller lift of $\det\bar\rho|_{I_v}$ by our choice of $\tilde K_v$. We conclude that π_v is an unramified principal series.

So we may assume all T/m appear in $W^P_{k_\lambda}$ and hence $\bar\rho^{\oplus\alpha} \simeq W^P_{k_\lambda}/mW^P_{k_\lambda}$ as a Galois-Hecke bimodule. For any lift π found in $W^P_{E_\lambda}$, π has an unramified special component at v. $q_vU(p_v)\cdot \mathrm{Fr}_v^{-1}$ is identity on $W^P_{E_\lambda}$ by the compatibility of local and global Langlands correspondence recalled in section 2 (by our normalization, the eigenvalue of $U(p_v)$-operator is equal to the Frobenius eigenvalue on $\rho_{\pi,\lambda}/\rho_{\pi,\lambda}^{I_v}$), and hence on $W^P_{\mathscr{O}_\lambda}$ also by the $\mathscr{O}_\lambda$-freeness. Since $U(p_v)$ commutes with global Galois action on $\bar\rho$, by Schur's lemma we may assume that there is a scalar $\gamma\in \bar k_\lambda\setminus\{0\}$ such that $\gamma^{-1}\mathrm{Fr}_v$ acts trivially on

$$W^P_{k_\lambda}/mW^P_{k_\lambda} = (H_{k_\lambda})^P_m/m(H_{k_\lambda})^P_m \otimes_{k_\lambda} \bar k_\lambda \simeq \bar\rho^{\oplus\alpha}.$$

It follows that any Fr_v-eigenvalues of $\bar\rho$ are the same, which leads to a contradiction since two eigenvalues of Fr_v on $\bar\rho$ are of the form $\bar\alpha_v,\ q_v\bar\alpha_v$, and $q_v\not\equiv 1 \mod \ell$.

5.2. **Cerednik-Drinfeld type theorem for totally real fields.** Let D be a quaternion algebra over F. Assume D defines a Shimura curve, i.e., $D\otimes_{\mathbb{Q}}\mathbb{R}\simeq M_2(\mathbb{R})\times \mathrm{H}^{g-1}$.

Choose a finite place v where $\mathrm{inv}_vD = 1/2$. Let $\tilde D$ be the definite quaternion with $\mathrm{inv}_vD = 0$, and other invariants at finite places are the same as D. We need a generalization of the Cerednik-Drinfeld theorem for Shimura curves over $\mathbb{Q}$. For general totally real fields, such a result follows from the work of Boutot-Zink [2], which we recall in the following.

We choose a compact open subgroup $K\subset D^\times(\mathbb{A}^\infty)$ such that $K = o^\times_{D_v}\cdot K^v$. Put $\tilde K = \mathrm{GL}_2(o_v)\cdot K^v$. Then the main result of [2] claims:

Theorem 5.3. *[[2], theorem 0.1] There is a canonical isomorphism*

$$S_{D,K,\ o_v} \simeq \tilde D^\times\backslash D^\times(\mathbb{A}^{v,\infty})\times\hat\Omega_{o_v}\hat\otimes_{o_v}o_v^{\mathrm{unr}}/K^v.$$

Here $\hat\Omega_{o_v}$ is the Deligne model of the Drinfeld upper half plane, and o_v^{unr} the maximal unramified extension of o_v. The action of $\mathrm{GL}_2(F_v)$ on $\hat\Omega_{o_v}\hat\otimes_{o_v}o_v^{\mathrm{unr}}$ is

$$g\mapsto (g,\ \mathrm{Fr}_v^{n(g)}),\quad n(g)=\mathrm{ord}_v(\det\, g),\ \mathrm{Fr}_v:\ \textit{Frobenius at } v$$

and $\tilde D^\times$ acts diagonally. Especially, $S_{D,K,\ o_v}$ is regular.

Corollary 5.4. *The set of irreducible components J_K of $S_{K,\ o_v^{\mathrm{unr}}}$ is identified with two copies of the double cosets $S_{\tilde D,\tilde K} = \tilde D^\times\backslash\tilde D^\times(\mathbb{A}^\infty)/\tilde K$, and the identification is $\tilde D^\times(\mathbb{A}^{v,\infty})$-equivariant.*

Proof. The set of irreducible components I of $\hat\Omega_{o_v}$ is isomorphic to the set of all lattices up to homothety in F_v^2 by the structure of the Bruhat-Tits building, and hence identified with $\mathrm{GL}_2(F_v)/F_v^\times\cdot\mathrm{GL}_2(o_v)$.

By theorem 5.3, J_K is canonically isomorphic to

$$\tilde D^\times\backslash D^\times(\mathbb{A}^{v,\infty})\times I\times\mathbb{Z}/K^v.$$

Put $\tilde D^\times_+ = \{g\in\tilde D^\times,\ \mathrm{ord}(\det g) \text{ is even}\}$.

The map $\alpha:\tilde D^\times_+\to 2\mathbb{Z}$ given by $\mathrm{ord}(\det g)$ is surjective. So the pieces $J_{K,+} = \tilde D^\times_+\backslash D^\times(\mathbb{A}^{v,\infty})\times I\times 2\mathbb{Z}/K^v$ and $J_{K,-} = \tilde D^\times_+\backslash D^\times(\mathbb{A}^{v,\infty})\times I\times(\mathbb{Z}\setminus 2\mathbb{Z})/K^v$ inside J_K are isomorphic to $\alpha^{-1}(0)\backslash\mathrm{GL}_2(F_v)/(F_v^\times\cdot\mathrm{GL}_2(o_v))/K^v\simeq S_{\tilde D,\tilde K}$. $\tilde D^\times(\mathbb{A}^{v,\infty})$-equivariance follows from the description. Moreover, the construction of $J_{K\pm}$ is canonical. □

We need to calculate the fibers of our sheaves $\mathscr{F}^D_{k,w}$ as well. Note that we also have a sheaf $\mathscr{F}^{\tilde D}_{k,w}$ which is an analogue of $\mathscr{F}^D_{k,w}$ on $S_{D,K}$.

Corollary 5.5. *The sum of geometric generic fibers $\oplus_{\eta\in J_K}(\bar{\mathscr{F}}_{k,w})_{\bar\eta}$ is identified with the sheaf $\mathscr{F}^{\tilde D}_{k,w}$ on $S_{\tilde D,\tilde K}$ on each $J_{K,\pm}$, and the identification is $\tilde D^\times(\mathbb{A}^{v,\ell,\infty})$-equivariant.*

Proof. Let $\pi_\ell : \tilde S_\ell \to S_{D,K}$ be the Galois covering corresponding to $\prod_{u|\ell} K_u/K\cap\overline{(F^\times)}$. By corollary 5.4, the set of irreducible components of $(\tilde S_\ell)_{\overline{k(v)}}$ is identified with two copies of $S_{\tilde D,\tilde K^\ell}$, preserving $\tilde D^\times(\mathbb{A}^{v,\infty})$-action. Since the sheaf $\mathscr{F}^D_{k,w}$ is obtained from the covering π_ℓ and the representation $\otimes_{\iota\in I_F}(\iota\det)^{k'_\iota}\operatorname{Sym}^{k_\iota-2}$ of $D^\times(\bar{\mathbb{Q}}_\ell)$ by contracted product, their geometric generic fibers are identified with $\mathscr{F}^{\tilde D}_{k,w}$ on each $J_{K,+}$ and $J_{K,-}$ by the $\tilde D^\times(\mathbb{A}^{v,\infty})$-equivariance. □

We need U_v-operator in the following, which is defined as follows. Take a uniformizer Π_v of $\mathscr{O}_{D_v}$, and consider the double coset $K^v\cdot K_v\Pi_vK_v$. This defines a correspondence, and the action on cohomology is U_v. U_v-operator thus defined exists over F, not only over F_v, and by the local Jacquet-Langlands correspondence [16], it corresponds to $U(p_v)$-operator defined in 5.1 for $\mathrm{GL}_2(F_v)$ (this is proved easily).

5.3. **The even degree case.** By 5.3 and the method of section 5.1, we prove 5.1 without assuming A-4). By subsection 5.1 and remark 4.7 b), we may assume that the degree $[F:\mathbb{Q}]$ is even, and π_v is a special representation twisted by an unramified character.

Take $\tilde D$, the definite quaternion algebra which is unramified at all finite places, and let D be an indefinite quaternion algebra corresponding to $\tilde D$ as above. By the argument of 4.1, we take an auxiliary place y, and take a compact open subgroup $K=\prod_u K_u$ with $K_v = o^\times_{D_v}$, $K_y = K_{11}(m_y)$. Then $\rho_{\pi,\lambda}$ occurs in $H^1_{\text{ét}}(S_{D,K},\bar{\mathscr{F}}^D_{k,w})$ by the Jacquet-Langlands correspondence, since π_v is an unramified special representation.

We choose Σ so that $u\notin\Sigma\Rightarrow u\not|\ell$ and $K_u=\mathrm{GL}_2(o_u)$. Let T be the Hecke algebra generated by $[T_u]$, $[T_{u,u}]$, $[T_{u,u}]^{-1}$ for $u\notin\Sigma$ over $\mathscr{O}_\lambda$, m be the maximal ideal of T corresponding to $\bar\rho$. Assume contrary, so $\bar\rho$ does not appear in $H^0(S_{\tilde D,\tilde K},\ \bar{\mathscr{F}}^{\tilde D}_{k,w}\otimes_{\mathscr{O}_\lambda}k_\lambda)$.

Proposition 5.6.

$$H^1_{\text{ét}}(S_{D,K,o_v^{\mathrm{unr}}},\mathscr{F}^D_{k,w}\otimes_{\mathscr{O}_\lambda}R)_m \simeq H^1_{\text{ét}}(S_{D,K,\bar F_v},\bar{\mathscr{F}}^D_{k,w}\otimes_{\mathscr{O}_\lambda}R)^{I_v}_m$$

holds for a finite local $\mathscr{O}_\lambda$-algebra R, and hence $R\mapsto H^1_{\text{ét}}(S_{D,K},\mathscr{F}^D_{k,w}\otimes_{\mathscr{O}_\lambda}R)^{I_v}_m$ commutes with scalar extensions.

Proof. We adopt the method used in previous sections. Since our T-action annihilates $(\oplus_{Y\in J}H^0_{\text{ét}}(Y,\mathscr{F}|_Y))_m\simeq H^0(S_{\tilde D,\tilde K},\ \mathscr{F}^{\tilde D}_{k,w}\otimes_{\mathscr{O}_\lambda}k_\lambda)^{\oplus 2}_m$ by our assumption that $\bar\rho$ does not come from $H^0(S_{\tilde D,\tilde K},\ \bar{\mathscr{F}}^{\tilde D}_{k,w}\otimes_{\mathscr{O}_\lambda}k_\lambda)$, the claim follows from 5.5 and 3.11. □

The rest of the argument is the same as in section 5.1. By proposition 5.6, $\bar\rho$ comes from $H^1_{\text{ét}}(S_{D,K,\bar F_v},\ \bar{\mathscr{F}}^{\tilde D}_{k,w})^{I_v}_m$ as a Hecke module. As in section 5.1, $q_vU_v\cdot\mathrm{Fr}_v^{-1}$ is identity on $H^1_{\text{ét}}(S_{D,K,\bar F_v},\ \bar{\mathscr{F}}^{\tilde D}_{k,w})^{I_v}$, and U_v-operator on $H^1_{\text{ét}}(S_{D,K,\bar F_v},\ \bar{\mathscr{F}}^{\tilde D}_{k,w}\otimes_{\mathscr{O}_\lambda}k_\lambda)^{I_v}_m\otimes_T T/m$ acts as a scalar since it commutes with the global Galois group G_F by using Boston-Lenstra-Ribet theorem[3]. Two eigenvalues of Fr_v on $\bar\rho$ are of the form $\bar\alpha_v,\ q_v\bar\alpha_v$, this implies $q_v\equiv 1$ mod ℓ.

6. Proof of theorem A

Now we finish the proof of theorem A in the introduction. We may assume that the degree $[F:\mathbb{Q}]$ is even. By [21], theorem 1, there is a finite place $z\neq v$ of F where $\bar\rho$ is unramified, $q_z\equiv -1 \mod \ell$, and there is a cuspidal representation $\tilde\pi$ which gives $\bar\rho$ such that $\tilde\pi_z$ is an unramified special representation at z, $(\tilde\pi^\infty)^{K\cap K_0(z)}\neq\{0\}$. We can now apply

theorem B to $\tilde{\pi}$, and optimize the level at v. When $\bar{\rho}$ is unramified at v, we first apply remark 4.7 b) to get π' which has an unramified special component π'_v at v, then apply the Mazur principle in the form of section 5.1. Finally, the auxiliary place z can be removed by the Mazur principle in 5.3, since $q_z \not\equiv 1 \mod \ell$, and the component at z is an unramified special representation.

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